SOME RESULTS ON LOCAL COHOMOLOGY MODULES

AMIR MAFI

Abstract. Let R be a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R, and let M be a finitely generated R-module. For a non-negative integer t, we prove that $H^t_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite whenever $H^t_{\mathfrak{a}}(M)$ is Artinian and $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i < t$. This result, in particular, characterizes the $\mathfrak{a}$-cofiniteness property of local cohomology modules of certain regular local rings. Also, we show that for a local ring $(R, \mathfrak{m})$, $f-\operatorname{depth}(\mathfrak{a}, M)$ is the least integer i such that $H^i_{\mathfrak{a}}(M) \ncong H^i_{\mathfrak{m}}(M)$. This result in conjunction with the first one, yields some interesting consequences. Finally, we extend the non- vanishing Grothendieck's Theorem to $\mathfrak{a}$-cofinite modules.

1. Introduction

Throughout this paper, we assume that R is a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R, and that M is an R-module. Let t be a non-negative integer. Grothendieck [**4**] introduced the local cohomology modules $H^t_{\mathfrak{a}}(M)$ of M with respect to $\mathfrak{a}$. He proved their basic properties. For example, for a finitely generated module M, he proved that $H^t_{\mathfrak{m}}(M)$ is Artinian for all t, whenever R is local with maximal ideal $\mathfrak{m}$. In particular, it is shown that $\operatorname{Hom}_R(R/\mathfrak{m}, H^t_{\mathfrak{m}}(M))$ is finitely generated. Later Grothendieck asked in [**5**] whether a similar statement is valid if $\mathfrak{m}$ is replaced by an arbitrary ideal. Hartshorne gave a counterexample in [**6**], where he also defined that an R-module M (not necessarily finitely generated) is $\mathfrak{a}$-cofinite, if $\operatorname{Supp}_R(M) \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^t_R(R/\mathfrak{a}, M)$ is a finitely generated R-module for all t. He also asked when the local cohomology modules are $\mathfrak{a}$-cofinite. In this regard, the best known result is that when either $\mathfrak{a}$ is principal or R is local and $\dim R/\mathfrak{a} = 1$, then the modules $H^t_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite. These results are proved in [**8**] and [**3**], respectively. Melkersson [**15**] characterized those Artinian modules which are $\mathfrak{a}$-cofinite. For a survey of recent developments on cofiniteness properties of local cohomology, see Melkersson's interesting article [**16**]. One of the aim of this note is to show that,

2000 *Mathematics Subject Classification.* 13D45.

Key words and phrases. local cohomology, cofinite.

for a finitely generated module M, the module $H^t_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite whenever the modules $H^i_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite for all $i < t$ and $H^t_{\mathfrak{a}}(M)$ is Artinian. This result, in particular, characterizes the $\mathfrak{a}$-cofiniteness property of local cohomology modules of certain regular local rings (see Remark 2.3(ii)). Next, we assume that R is local with maximal ideal $\mathfrak{m}$. We prove that $f-\operatorname{depth}(\mathfrak{a}, M)$, which was introduced in [**14**], is the least integer i such that $H^i_{\mathfrak{a}}(M) \ncong H^i_{\mathfrak{m}}(M)$. This result together with our first mentioned result, in turn yields some interesting consequences. Finally, we extend the non-vanishing Grothendieck's Theorem for $\mathfrak{a}$-cofinite R-modules.

2. The results

The following theorem describes the behaviour of the cofiniteness and Artinian property on local cohomology modules.

Theorem 2.1. *Let M be finitely generated such that $H^t_{\mathfrak{a}}(M)$ is Artinian and that $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i < t$. Then $H^t_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite.*

Proof. In view of [**16**, Proposition 4.1], it is enough to prove that $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ is of finite length. To prove this, by [**18**, Theorem 11.38], we consider the Grothendieck spectral sequence

$$E_2^{i,j} = \operatorname{Ext}^i_R(R/\mathfrak{a}, H^j_{\mathfrak{a}}(M)) \underset{i}{\Longrightarrow} \operatorname{Ext}^{i+j}_R(R/\mathfrak{a}, M).$$

Since $E_r^{0,t} \cong E_\infty^{0,t}$ for r sufficiently large, $E_\infty^{0,t}$ is isomorphic to a subquotient of $\operatorname{Ext}^t_R(R/\mathfrak{a}, M)$ and, furthermore, $\ker d_{r-1}^{0,t} \cong E_\infty^{0,t}$ for all $r \geq 3$, where $\ker d_{r-1}^{0,t} = \ker(E_{r-1}^{0,t} \longrightarrow E_{r-1}^{r-1,t-r+2})$, we can deduce that $\ker d_{r-1}^{0,t}$ is finitely generated for r sufficiently large. Next, for all $r \geq 3$, we have the exact sequence

$$0 \longrightarrow \ker d_{r-1}^{0,t} \longrightarrow E_{r-1}^{0,t} \longrightarrow E_{r-1}^{r-1,t-r+2}.$$

Therefore, since $E_{r-1}^{r-1,t-r+2}$ is a subquotient of $E_2^{r-1,t-r+2}$, our hypothesis give us that $E_{r-1}^{0,t}$ is finitely generated for r sufficiently large. continuing in this fashion, we see that $E_2^{0,t}$ is finitely generated; and hence it is of finite length. $\square$

The following corollary is immediate.

Corollary 2.2. *Let M be finitely generated. Suppose that the local cohomology module $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i < t$ and that it is Artinian for all $i \geq t$. Then $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all i.*

Remarks 2.3. (i) There is an example in [**7**, Example 3.4] which shows that $H^t_{\mathfrak{a}}(R)$ is not $\mathfrak{a}$-cofinite for $t = \operatorname{grade}(\mathfrak{a})$. However, by the above Theorem, $H^t_{\mathfrak{a}}(R)$ is $\mathfrak{a}$-cofinite, whenever it is Artinian.
(ii) Let $(R, \mathfrak{m})$ be a regular local ring of characteristic $p(> 0)$ and of dimension n. Suppose that $R/\mathfrak{a}$ is a generalized Cohen-Macaulay local ring of dimension $d(> 0)$. Then, by [**20**, Corollary 1.7] and Theorem 2.1, the local cohomology modules $H^i_{\mathfrak{a}}(R)$ are $\mathfrak{a}$-cofinite if and only if $H^{n-d}_{\mathfrak{a}}(R)$ is $\mathfrak{a}$-cofinite.

Let R be a local ring with maximal ideal $\mathfrak{m}$ and let M be a finitely generated. Following [**9**], a sequence $x_1, \ldots, x_n$ of elements of R is said to be an M-filter regular sequence if, for all $\mathfrak{p} \in \operatorname{Supp}(M)\backslash\{\mathfrak{m}\}$, the sequence $x_1/1, \ldots, x_n/1$ of elements of $R_{\mathfrak{p}}$ is a poor $M_{\mathfrak{p}}$-regular sequence. For an ideal $\mathfrak{a}$ of R, the $f-\operatorname{depth}$ of $\mathfrak{a}$ on M is defined as the length of any maximal M-filter regular sequence in $\mathfrak{a}$, denoted by $f - \operatorname{depth}(\mathfrak{a}, M)$. Here, when a maximal M-filter regular sequence in $\mathfrak{a}$ does not exist, we understand that the length is ∞. For some basic applications of these sequences see [**2**].

Lemma 2.4. *Let $(R, \mathfrak{m})$ be a local ring and suppose that M is finitely generated. Then $f - \operatorname{depth}(\mathfrak{a}, M) = \min\{i \in \mathbb{N}_0 : \operatorname{Supp}_R H^i_{\mathfrak{a}}(M) \nsubseteq \{\mathfrak{m}\}\}$.*

Proof. Let $x_1, \ldots, x_n$ be a maximal M-filter regular sequence in $\mathfrak{a}$. If there exists $\mathfrak{p} \in \operatorname{Supp}_R(H^i_{\mathfrak{a}}(M))\backslash\{\mathfrak{m}\}$ for some $0 \leq i \leq n-1$, then $x_1/1, \ldots, x_n/1$ is an $M_{\mathfrak{p}}$-regular sequence contained in $\mathfrak{a}R_{\mathfrak{p}}$. Hence $H^i_{\mathfrak{a}}(M)_{\mathfrak{p}} = 0$, which is a contradiction. It therefore follows that

$$f - \operatorname{depth}(\mathfrak{a}, M) \leq \min\{i \in \mathbb{N}_0 : \operatorname{Supp}_R H^i_{\mathfrak{a}}(M) \nsubseteq \{\mathfrak{m}\}\}.$$

Next, by assumption on $x_1, \ldots, x_n$, there exists $\mathfrak{p} \in \operatorname{Ass}_R(M/(x_1, \ldots, x_n)M)\backslash\{\mathfrak{m}\}$with $\mathfrak{a} \subseteq \mathfrak{p}$. Now $\mathfrak{p} \in \operatorname{Ass}_R(\operatorname{Hom}_R(R/\mathfrak{a}, M/(x_1, \ldots, x_n)M))$; and hence $\mathfrak{p} \in \operatorname{Ass}_R(\operatorname{Ext}^n_R(R/\mathfrak{a}, M))\backslash\{\mathfrak{m}\}$. Therefore, by [**11**, Proposition 1.1], $\mathfrak{p} \in \operatorname{Supp}(H^n_{\mathfrak{a}}(M))\backslash\{\mathfrak{m}\}$, and this completes the proof. □

Theorem 2.5. *(see [**9**, Theorem 3.10] and [**14**, Theorem 3.1]) Let $(R, \mathfrak{m})$ be a local ring and suppose that M is finitely generated. Then $f - \operatorname{depth}(\mathfrak{a}, M) = \min\{i \in \mathbb{N}_0 : H^i_{\mathfrak{a}}(M) \ncong H^i_{\mathfrak{m}}(M)\}$.*

Proof. If $\mathrm{Supp}_R(M/\mathfrak{a}M) \subseteq \{\mathfrak{m}\}$, then $\sqrt{\mathfrak{a} + Ann(M)} = \mathfrak{m}$; and hence $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{m}}(M)$ for all $i \geq 0$. Therefore $\min\{i \in \mathbb{N}_0 : H^i_{\mathfrak{a}}(M) \ncong H^i_{\mathfrak{m}}(M)\} = \infty = f - \mathrm{depth}(\mathfrak{a}, M)$; and the result follows. So, we may assume that $\mathrm{Supp}_R(M/\mathfrak{a}M) \nsubseteq \{\mathfrak{m}\}$. Let $t = f - \mathrm{depth}(\mathfrak{a}, M)$ and let $x_1, \ldots, x_t$ be an M-filter regular sequence in $\mathfrak{a}$. Then, by [**19**, Lemma 1.19], $H^i_{\mathfrak{a}}(M) \cong H^i_{(x_1,\ldots,x_t)}(M) \cong H^i_{\mathfrak{m}}(M)$, for all $i < t$. On the other hand, by Lemma 2.4, the R-module $H^t_{\mathfrak{a}}(M)$ is not isomorphic with $H^t_{\mathfrak{m}}(M)$. It therefore follows, by [**9**, Theorem 3.10]. □

Remarks 2.6. Let M be finitely generated. Then
(i) in view of Theorem 2.1 and Theorem 2.5, it is clear that if $(R, \mathfrak{m})$ is a local ring, then $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all i less than $f - \mathrm{depth}(\mathfrak{a}, M)$;
(ii) it follows immediately from [**9**, Theorem 3.10] and Theorem 2.5 that if $(R, \mathfrak{m})$ is local and $H^i_{\mathfrak{a}}(M)$ is Artinian for all $i < t$, then $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{m}}(M)$ for all $i < t$.

The following lemma is needed in the proof of the next theorem. Note that if we replace $\mathfrak{a}$ by the zero ideal in the lemma, then the Grothendieck's Theorem [**4**, p.88] immediately follows.

Lemma 2.7. *Let M be $\mathfrak{a}$-cofinite. Then for every maximal ideal $\mathfrak{m}$ of R and for all t, $H^t_{\mathfrak{m}}(M)$ is Artinian.*

Proof. Since $H^t_{\mathfrak{m}}(M)$ is an $\mathfrak{a}$-torsion module, by [**13**, Theorem 1.3], it is enough to prove $0 :_{H^t_{\mathfrak{m}}(M)} \mathfrak{a}$ is Artinian. Let $\Phi(-)$ denote the composite functor $\mathrm{Hom}_R(R/\mathfrak{a}, H^0_{\mathfrak{m}}(-))$. We get a spectral sequence arising from the composite functor as:

$$E_2^{i,j} = \mathrm{Ext}^i_R(R/\mathfrak{a}, H^j_{\mathfrak{m}}(M)) \Longrightarrow (R^{i+j}\Phi)(M).$$

Now, we use induction on j (with $0 \leq j \leq t$) to show that $E_2^{0,t}$ is Artinian. Let $0 \leq j < t$ and suppose that the result has been proved for smaller values of j.(Note that the case $j = 0$ was proved in [**15**, Corollary 1.8].) We can apply [**15**, Theorem 1.9] and use a similar argument as in the proof of Theorem 2.1, to see that $\ker d_{r-1}^{0,j+1}$ is Artinian for r sufficiently large. On the other hand, by induction, $E_{r-1}^{r-1,j-r+3}$ is

Artinian. It now follows that $E_2^{0,j+1}$ is Artinian. This complete the inductive step. In particular $E_2^{0,t}$ is Artinian. $\square$

In the next result, we will use the concept of attached prime ideals. For more details in this subject the reader is referred to [**10**] or the appendix to §**6** in [**12**].

Theorem 2.8. *Let $(R, \mathfrak{m})$ be a local ring and let M be a module of dimension d. If $H^d_{\mathfrak{m}}(M)$ is an Artinian module, then if $\mathfrak{p}$ is any of its attached prime ideals, one has $\dim R/\mathfrak{p} \geq d$.*

Proof. From the right exactness of $H^d_{\mathfrak{m}}(-)$ on modules of dimension $\leq d$, we get $H^d_{\mathfrak{m}}(M/\mathfrak{p}M) \cong H^d_{\mathfrak{m}}(M)/\mathfrak{p}H^d_{\mathfrak{m}}(M)$, which is $\neq 0$, since $\mathfrak{p}$ is an attached prime ideal of $H^d_{\mathfrak{m}}(M)$. But $M/\mathfrak{p}M$ is a module over $R/\mathfrak{p}$. Therefore $\dim R/\mathfrak{p} \geq d$. $\square$

In the following theorem, which establishes the non-vanishing Grothendeick Theorem for $\mathfrak{a}$-cofinite modules.

Theorem 2.9. *Let $(R, \mathfrak{m})$ be a local ring and let M be a non-zero $\mathfrak{a}$-cofinite R-module of dimension n. Then $H^n_{\mathfrak{m}}(M) \neq 0$.*

Proof. Firstly note that, in view of the hypotheses, $0 :_M \mathfrak{a}$ is a finitely generated R-module of dimension n. Now, we prove the theorem by induction on $n (\geq 0)$. If $n = 0$, then $0 :_M \mathfrak{a}$ is Artinian; and hence, by [**13**, Theorem 1.3], M is Artinian. Therefore $H^0_{\mathfrak{m}}(M) = M \neq 0$.
Suppose, inductively, that $n \geq 1$ and the result has been proved for $n-1$. We may assume that M is $\mathfrak{m}$-torsion free. Also, by [**15**, Corollary 1.4], we may assume that $\mathrm{Ass}(M)$ is a finite set. Then, there exists a non-zero divisor $x \in \mathfrak{m}$ on M. Suppose the contrary that $H^n_{\mathfrak{m}}(M) = 0$. Then, for any such x, we can consider the exact sequence $0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$ to see that $H^{n-1}_{\mathfrak{m}}(M)/xH^{n-1}_{\mathfrak{m}}(M) \cong H^{n-1}_{\mathfrak{m}}(M/xM)$,
$n - 1 = \dim(0 :_M \mathfrak{a})/x(0 :_M \mathfrak{a}) \leq \dim(0 :_{M/xM} \mathfrak{a}) = \dim M/xM \leq n-1$, and that, by [**15**, Remark(a)], M/xM is $\mathfrak{a}$-cofinite. Therefore, by induction hypothesis, $H^{n-1}_{\mathfrak{m}}(M)/xH^{n-1}_{\mathfrak{m}}(M) \neq 0$. Note that, by Lemma 2.7, $H^{n-1}_{\mathfrak{m}}(M)$ is Artinian. If $\mathfrak{m} \notin \mathrm{Att}\, H^{n-1}_{\mathfrak{m}}(M)$, then, for any

$$y \in \mathfrak{m} \setminus \bigcup_{\mathfrak{p} \in \mathrm{Att}\, H^{n-1}_{\mathfrak{m}}(M)} \mathfrak{p} \bigcup \bigcup_{\mathfrak{q} \in \mathrm{Ass}(M)} \mathfrak{q},$$

we have $H^{n-1}_{\mathfrak{m}}(M) = yH^{n-1}_{\mathfrak{m}}(M)$, which is a contradiction. Thus $\mathfrak{m} \in \operatorname{Att} H^{n-1}_{\mathfrak{m}}(M)$. Let $\operatorname{Att} H^{n-1}_{\mathfrak{m}}(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t, \mathfrak{m}\}$ and let $z \in \mathfrak{m} \setminus \bigcup_{i=1}^{t} \mathfrak{p}_i \bigcup \bigcup_{\mathfrak{q} \in \operatorname{Ass}(M)} \mathfrak{q}$. Then, by the above argument, we have $H^{n-1}_{\mathfrak{m}}(M)/zH^{n-1}_{\mathfrak{m}}(M) \cong H^{n-1}_{\mathfrak{m}}(M/zM)$. Hence, by [**17**, Proposition 5.2], $\operatorname{Att} H^{n-1}_{\mathfrak{m}}(M/zM) = \operatorname{Supp}(R/(zR)) \cap \operatorname{Att} H^{n-1}_{\mathfrak{m}}(M) = \{\mathfrak{m}\}$. Therefore, by [**1**, Corollary 7.2.12], $H^{n-1}_{\mathfrak{m}}(M/zM)$ has finite length. If we show that $H^{n-1}_{\mathfrak{m}}(M/zM) = 0$, then we achieved at the required contradiction. To this end, first let $n = 1$. Then we have the exact sequence

$$0 \to H^0_{\mathfrak{m}}(M) \xrightarrow{z} H^0_{\mathfrak{m}}(M) \to H^0_{\mathfrak{m}}(M/zM) \to H^1_{\mathfrak{m}}(M).$$

By our hypothesis $H^0_{\mathfrak{m}}(M) = 0 = H^1_{\mathfrak{m}}(M)$; and so $H^0_{\mathfrak{m}}(M/zM) = 0$. Now, we assume that $n > 1$. Then, Theorem 2.8 implies that attached prime ideals of $H^{n-1}_{\mathfrak{m}}(M/zM)$ is empty; and so $H^{n-1}_{\mathfrak{m}}(M/zM) = 0$. $\square$

Acknowledgment: The author is deeply grateful to the referee for his or her careful reading of the manuscript. The author also thank the referee for proposing Theorem 2.8.

References

[1] M. P. Brodmann and R. Y. Sharp, *Local cohomology-an algebric introduction with geometric applications*, Cambridge University Press, Cambridge, 1998.

[2] N. T. Cuong, P. Schenzel and N. V. Trung, *Verallgemeinerte Cohen- Macaulay- Moduln*, Math. Nachr. **85**(1978), 57-73.

[3] D. Delfino and T. Marley, *Cofinite modules and local cohomology*, J. Pure Appl. Alg. **121**(1997), 45-52.

[4] A. Grothendieck, *Local cohomology*, Lecture Notes in Math., vol 41, Springer, Berlin, 1967.

[5] A. Grothendieck, *Cohomologie locale des faisceaux et théorèmes Lefshetz cohérents locaux et globaux*, Noth-Holland, Amsterdam, 1968.

[6] R. Hartshorne, *Affine duality and cofiniteness*, Inven. Math. **9**(1970), 145-164.

[7] C. Huneke and J. Koh, *Cofiniteness and vanishing of local cohomology modules*, Math. Proc.Camb. Phil. Soc. **110**(1991), 421-429.

[8] K. I. Kawasaki, *Cofiniteness of local cohomology modules for principle ideals*, Bull. London. Math. Soc. **30**(1998), 241-246.

[9] R. Lü and Z. Tang, *The f-depth of an ideal on a module*, Proc. Amer. Math. Soc. **130**(7)(2001), 1905-1912.

[10] I. G. Macdonald, *Secondary representations of modules over a commutative ring*, Sympos. Math., vol. **11**, Academic Press, London and New York, 1973, pp. 23-43.

[11] T. Marley, *The associated primes of local cohomology modules over rings of small dimension*, Manuscripta Math. **104**(4)(2001), 519-525.

[12] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.

[13] L. Melkersson, *On asymptotic stability for sets of prime ideals connected with the powers of an ideal*, Math. Proc. Camb. Phil. Soc. **107**(1990), 267-271.

[14] L. Melkersson, *Some applications of a criterion for Artinianness of a module*, J. Pure Appl. Alg. **101**(3)(1995), 291-303.

[15] L. Melkersson, *Properties of cofinite modules and applications to local cohomology*, Math. Proc. Camb. Phil. Soc. **125**(3)(1999), 417-423.

[16] L. Melkersson, *Modules cofinite with respect to an ideal*, J. Alg. **285**(2)(2005),649-668.

[17] L. Melkersson and P. Schenzel, *The co-localization of an artinian module*, Proc. Edinburgh Math. Soc. **38**(2)(1995), 121-132.

[18] J. Rotman, *An introduction to homological algebra*, Academic Press, Orlando, 1979.

[19] P. Schenzel, *On the use of local cohomology in algebra and geometry*, Lectures at the summer school of commutative algebra and algebraic geometry, Ballaterra, 1996, Brikhä user. Verlag, 1998.

[20] R. Y. Sharp, *The Frobenius homomorphism, and local cohomology in regular local rings of positive characteristic*, J. Pure Appl. Alg. **71**(1991), 313-317.

A. Mafi, Arak University, Beheshti St., P.O. Box:879, Arak, Iran

E-mail address: a-mafi@araku.ac.ir